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Antenna subtraction at NNLO with hadronic initial states: real-virtual initial-initial configurations

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Abstract: The antenna subtraction method handles real radiation contributions in higher order corrections to jet observables. The method is based on antenna functions, which encapsulate all unresolved radiation between a pair of hard radiator partons. To apply this method to compute hadron collider observables, initial-initial antenna functions with both radiators in the initial state are required. In view of extending the antenna subtraction method to next-to-next-to-leading order (NNLO) calculations at hadron colliders, we derive the one-loop initial-initial antenna functions in unintegrated and integrated form.

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Contents

1 Introduction

Jet production observables are studied extensively at hadron colliders. Since the distribution of final state jets relates directly to the parton-level dynamics, jet observables can be used for precision studies of QCD [1, 2], especially in view of determinations of the strong coupling constant and the parton distribution functions in the proton. Experimental measurements of these observables at the Tevatron [3–5] attained an accuracy of a few per cent (or even better in certain kinematical ranges), and first results from the LHC [6–8] already show the potential for precision jet physics. Consequently, meaningful precision studies must rely on theoretical predictions accurate to the same level. In perturbative QCD, this precision usually requires corrections at next-to-next-to-leading order (NNLO).

NNLO calculations of observables with n jets in the final state require several ingredients: the two-loop corrected n-parton matrix elements, the one-loop corrected $(n+1)$-parton matrix elements, and the tree-level $(n+2)$-parton matrix elements. For most massless jet observables of phenomenological interest, these matrix elements are available for some time already.

The $(n+1)$-parton and $(n+2)$-parton matrix elements contribute to n jet observables at NNLO if the extra partons are unresolved or are clustered to form an n-jet final state. Consequently, these extra partons are unconstrained in the soft and collinear regions, and yield infrared divergences. In these cases, the infrared singular parts of the matrix elements need to be extracted and integrated over the phase space appropriate to the unresolved configuration to make the infrared pole structure explicit. The single soft and collinear limits of one-loop matrix elements [9–19] and the double unresolved limits of tree-level matrix elements [20–26] are process-independent, and result in a factorization into an unresolved factor times a matrix element of lower multiplicity.

To determine the contribution to NNLO jet observables from these configurations, one has to find subtraction terms which coincide with the full matrix element and are still sufficiently simple to be integrated analytically in order to cancel their infrared pole structure with the two-loop virtual contribution. Often starting from systematic methods for subtraction at NLO [27–33], several NNLO subtraction methods have been proposed in the literature [34–48], and are worked out to a varying level of sophistication.

For observables with partons only in the final state, an NNLO subtraction formalism, antenna subtraction, has been derived in [49]. The antenna subtraction formalism constructs the subtraction terms from antenna functions. Each antenna function encapsulates all singular limits due to the emission of one or two unresolved partons between two colour-connected hard radiator partons. This construction exploits the universal factorization of matrix elements and phase space in all unresolved limits. The antenna functions are derived systematically from physical matrix elements [50–52]. This formalism has been applied in the derivation of NNLO corrections to three-jet production in electron-positron annihilation [53–57] and related event shapes [58–62], which were used subsequently in precision determinations of the strong coupling constant [63–72]. The formalism can be extended to include parton showers at higher orders [73, 74], thereby offering a process-independent matching of fixed-order calculations and logarithmic resummations [66–68, 75, 76], which is done on a case-by-case basis for individual observables [77] up to now. The formalism can be extended to include massive fermions [78, 79].

For processes with initial-state partons, antenna subtraction has been fully worked out only to NLO so far [80]. In this case, one encounters two new types of antenna functions, initial-final antenna functions with one radiator parton in the initial state, and initial-initial antenna functions with both radiator partons in the initial state. The framework for the construction of NNLO antenna subtraction terms involving one or two partons in the initial state has been set up in [81] in the context of a proof-of-principle implementation of the contribution of the $gg \to 4g$ tree-level subprocess to di-jet production at hadron colliders. The initial-final and initial-initial antenna functions appearing in the NNLO subtraction terms are obtained from crossing the final-final antennae. Their integration has to be performed over the appropriate phase space. In the case of the initial-final antennae, this has been accomplished in [82]. For the initial-initial tree-level double real radiation antenna functions, partial results have been obtained in [83]. It is the aim of the present paper to derive the setup for NNLO antenna subtraction for single unresolved singularities at one-loop and to compute the integrated one-loop initial-initial antenna functions required in this context.

Other approaches to perform NNLO calculations of exclusive observables with initial state partons are the use of sector decomposition and a subtraction method based on the transverse momentum structure of the final state. The sector decomposition algorithm [84–87] analytically decomposes both phase space and loop integrals into their Laurent expansion in dimensional regularization, and performs a subsequent numerical computation of the coefficients of this expansion. Using this formalism, NNLO results were obtained for Higgs production [88–90] and vector boson production [91] at hadron colliders. Both reactions were equally computed independently [92–94] using an NNLO subtraction formalism exploiting the specific transverse momentum structure of these observables [45], which was also applied most recently to compute NNLO corrections to associated WH production [95].

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This paper is structured as follows: in section 2, we construct the subtraction terms required at NNLO for initial-initial configurations with one unresolved parton at one loop. They require one-loop $2 \to 2$ antenna functions with two partons in the initial state and one parton and one off-shell neutral current in the final state. The analytic integration of the initial-initial one-loop antenna functions is described in section 3. Finally, we conclude with an outlook in section 4.

2 Initial-intial antenna subtraction at NNLO

Antenna subtraction of initial-initial configurations at NLO is derived in detail in [80]. Subtraction terms with two hard partons in the initial state are built along the same lines as in the final-final and initial-final case. The NLO antenna subtraction term for an m-jet production process, to be convoluted with the appropriate parton distribution functions for the initial state partons, for a configuration with the two hard emitters in the initial state (partons i and k with momenta p_1 and p_2), reads:

$$\begin{aligned} d\hat{\sigma}^{S,(II)} = \mathcal{N} \sum_{m+1} & \mathrm{d}\Phi_{m+1}(k_1,\ldots,k_{j-1},k_j,k_{j+1},\ldots,k_{m+1};p_1,p_2)\,\frac{1}{S_{m+1}} \\ & \sum_j X^0_{ik,j}(p_1,p_2,k_j)\left|\mathcal{M}_m(\tilde{k}_1,\ldots,\tilde{k}_{j-1},\tilde{k}_{j+1},\ldots,\tilde{k}_{m+1};x_1p_1,x_2p_2)\right|^2 \\ & \times J^{(m)}_m(\tilde{k}_1,\ldots,\tilde{k}_{j-1},\tilde{k}_{j+1},\ldots,\tilde{k}_{m+1})\,. \end{aligned} \tag{2.1}$$

All the momenta in the arguments of the reduced matrix elements and the jet functions are redefined, which is a consequence of requiring the correct collinear factorization properties in both initial-state collinear limits. It should be noted that the jet function $J^{(m)}_m$ (constructing m jets from m partons) requires all redefined momenta to be resolved. The two hard radiators are simply rescaled by factors x_1 and x_2 respectively. The spectator momenta are boosted by a Lorentz transformation onto the new set of momenta $\{\tilde{k}_l,\, l \neq j\}$. The mapping must be based on a factorization of the $(m+1)$-particle phase space, must satisfy overall momentum conservation and keep the mapped momenta on the mass shell. In this case, this turns out to severely restrict the possible mappings.

The tree-level antenna function $X^0_{ik,j}$ depends only on the incoming momenta p_1, p_2 and on the outgoing momentum k_j. It accounts for all singular configurations where parton j is unresolved and colour-connected to partons i (incoming with momentum p_1) and k (incoming with momentum p_2). The jet function $J^{(m)}_m$ and the reduced matrix element in (2.1) depend only on the redefined momenta. With a suitable factorization of the phase space [80], one can perform the integration of the antenna function analytically.

The factorization of the phase space is obtained by requiring that the two mapped initial state momenta should be of the form

$$P_1 = x_1 p_1 \qquad P_2 = x_2 p_2\,, \tag{2.2}$$

so that

$$\tilde{q} \equiv P_1 + P_2$$

is in the beam axis. Since the vector component of $q \equiv p_1 + p_2 - k_j$ is in general not along the $p_1 - p_2$ axis we need to boost all the momenta so that $\tilde{q} = \Lambda q$ and $\tilde{k}_l = \Lambda k_l$ in order to restore momentum conservation. By requiring this boost to be only transverse, the phase space mapping is determined uniquely, resulting in the factorization

$$\begin{aligned} \mathrm{d}\Phi_{m+1}(k_1,\ldots,k_{m+1};p_1,p_2) = \mathrm{d}\Phi_m(\tilde{k}_1,\ldots,\tilde{k}_{j-1},\tilde{k}_{j+1},\ldots,\tilde{k}_{m+1};x_1p_1,x_2p_2) \\ \times\delta(x_1-\hat{x}_1)\,\delta(x_2-\hat{x}_2)\,[\mathrm{d}k_j]\,\mathrm{d}x_1\,\mathrm{d}x_2\,. \end{aligned} \tag{2.3}$$

where $[\mathrm{d}k] = \mathrm{d}^d k \delta^{(+)}(k^2)/(2\pi)^{d-1}$ and

$$\begin{aligned} \hat{x}_1 &= \left(\frac{s_{12}-s_{j2}}{s_{12}}\,\frac{s_{12}-s_{1j}-s_{j2}}{s_{12}-s_{1j}}\right)^{\frac{1}{2}}\,, \\ \hat{x}_2 &= \left(\frac{s_{12}-s_{1j}}{s_{12}}\,\frac{s_{12}-s_{1j}-s_{j2}}{s_{12}-s_{j2}}\right)^{\frac{1}{2}}\,. \end{aligned} \tag{2.4}$$

Inserting the factorized expression for the phase space measure in (2.1), the subtraction terms can be integrated over the antenna phase space. In the case of initial-initial subtraction terms, the antenna phase space is trivial: the two remaining Dirac delta functions can be combined with the one particle phase space, such that there are no integrals left. We define the initial-initial integrated antenna functions as follows:

$$\mathcal{X}_{ik,j}(x_1,x_2) = \frac{1}{C(\epsilon)}\int [\mathrm{d}k_j]\,x_1\,x_2\,\delta(x_1-\hat{x}_1)\,\delta(x_2-\hat{x}_2)\,X_{ik,j}, \tag{2.5}$$

where we introduced $C(\epsilon) = (4\pi)^\epsilon/(8\pi^2)e^{-\gamma_E\epsilon}$.
Substituting the one-particle phase space, and carrying out the integrations over the Dirac delta functions, we have,

$$\mathcal{X}_{ik,j}(x_1,x_2) = (Q^2)^{-\epsilon}\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\,\mathcal{J}(x_1,x_2)\,Q^2\,X_{ik,j}\,, \tag{2.6}$$

with $Q^2 = q^2 = (p_1+p_2-k_j)^2$. The Jacobian factor, $\mathcal{J}(x_1,x_2)$ is given by

$$\mathcal{J}(x_1,x_2) = \frac{x_1\,x_2\,(1+x_1\,x_2)}{(x_1+x_2)^2}\,(1-x_1)^{-\epsilon}(1-x_2)^{-\epsilon}\left(\frac{(1+x_1)(1+x_2)}{(x_1+x_2)^2}\right)^{-\epsilon}\,, \tag{2.7}$$

and the two-particle invariants are given by:

$$s_{1j} = -s_{12}\frac{x_1\,(1-x_2^2)}{x_1+x_2}\,,\qquad s_{j2} = -s_{12}\frac{x_2\,(1-x_1^2)}{x_1+x_2}\,. \tag{2.8}$$

The integrated subtraction term is then,

$$\begin{aligned} \mathrm{d}\hat{\sigma}^{S,(II)} = \sum_{m+1}\sum_j \frac{\mathcal{N}}{S_{m+1}}\int\frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2}\,C(\epsilon)\,\mathcal{X}_{ik,j}(x_1,x_2) \\ \times\mathrm{d}\Phi_m(k_1,\ldots,k_{j-1},k_{j+1},\ldots,k_{m+1};x_1p_1,x_2p_2) \\ \times|\mathcal{M}_m(k_1,\ldots,k_{j_1},k_{j+1},\ldots,k_{m+1};x_1p_1,x_2p_2)|^2 \\ \times J_m^{(m)}(k_1,\ldots,k_{j-1},k_{j+1},\ldots,k_{m+1})\,, \end{aligned} \tag{2.9}$$

where we have relabeled all $\tilde{k}_i \to k_i$. The final step is to convolute this subtraction term with the parton distribution functions of the initial state particles. The integrated version of the subtraction pieces is then combined with the virtual and mass factorization terms to yield a finite contribution when $\epsilon \to 0$. Recasting the convolutions appropriately, the integrated subtraction term is

$$\begin{aligned}
\mathrm{d}\hat{\sigma}^{S,(II)} = & \sum_{m+1} \sum_j \frac{S_m}{S_{m+1}} \int \frac{\mathrm{d}\xi_1}{\xi_1} \int \frac{\mathrm{d}\xi_2}{\xi_2} \int_{\xi_1}^{1} \frac{\mathrm{d}x_1}{x_1} \int_{\xi_2}^{1} \frac{\mathrm{d}x_2}{x_2} f_{i/1}\left(\frac{\xi_1}{x_2}\right) f_{k/2}\left(\frac{\xi_2}{x_2}\right) \\
& \times C(\epsilon)\, \mathcal{X}_{ik,j}(x_1, x_2)\, \mathrm{d}\hat{\sigma}^B(\xi_1 H_1, \xi_2 H_2)\,.
\end{aligned} \tag{2.10}$$

This convolution already has the appropriate structure and combination with the virtual corrections and mass factorization can be carried out explicitly leaving a finite contribution. The remaining phase space integration, implicit in the Born cross section, $\mathrm{d}\hat{\sigma}^B$, and the convolutions can be safely evaluated numerically.

At NNLO, two types of contributions to m-jet observables require subtraction: the tree-level $(m+2)$-parton matrix elements (where one or two partons can become unresolved), and the one-loop $(m+1)$-parton matrix elements (where one parton can become unresolved). The corresponding subtraction terms are denoted by $\mathrm{d}\hat{\sigma}^{S}_{\mathrm{NNLO}}$ and $\mathrm{d}\hat{\sigma}^{\mathrm{VS},1}_{\mathrm{NNLO}}$. Antenna subtraction terms for the final-final [49] and initial-final [82] cases have been derived previously. In the initial-initial case, $\mathrm{d}\hat{\sigma}^{S}_{\mathrm{NNLO}}$ was derived in [81, 83]. It contains subtraction terms for single unresolved limits (each containing a single three-particle antenna function X_3^0) and for double unresolved limits (containing four particle antenna functions X_4^0, products of three-particle antenna functions $(X_3^0 \cdot X_3^0)$ and soft large-angle correction terms $(S \cdot X_3^0)$).

The integrand in the $(m+1)$-parton channel consists, besides the one-loop $(m+1)$-parton matrix elements, of several contributions (independent of whether the radiators are in the initial or final state):

(a) The integrated one-particle unresolved subtraction terms from the $(m+2)$-parton channel, which cancel the explicit infrared poles of the virtual one-loop $(m+1)$-parton matrix element.

(b) The virtual-unresolved subtraction term $\mathrm{d}\hat{\sigma}^{\mathrm{VS},1,b}_{\mathrm{NNLO}}$ which subtracts all single unresolved limits from the virtual one-loop $(m+1)$-parton matrix element.

(c) Terms common to both above contributions, which are oversubtracted. Each of these terms is formed by a product of an integrated and an unintegrated three-parton tree-level antenna function $(\mathcal{X}_3^0 \cdot X_3^0)$. These terms contain the full set of singly integrated $(X_3^0 \cdot X_3^0)$-terms from $\mathrm{d}\hat{\sigma}^{S}_{\mathrm{NNLO}}$, plus additional terms which must be further integrated down to the m-parton channel.

(d) The integrated soft large-angle correction terms $(\mathcal{S} \cdot X_3^0)$.

(e) Terms arising from the mass factorization of the parton distribution functions at NLO.

Unintegrated subtraction terms newly introduced in the $(m+1)$-parton channel have to be compensated by their integrated forms in the m-parton channel. The integration of contributions of type (b) in initial-initial kinematics is the main topic of this paper, they are already known for final-final [49] and initial-final [82] kinematics. It should be noted that integration of terms of type (c) does not require any new integrals beyond the $\mathcal{X}_3^0$ already needed at NLO [80]. In particular, those terms obtained by integrating $(X_3^0 \cdot X_3^0)$ from $\mathrm{d}\hat{\sigma}^S_{\mathrm{NNLO}}$ depend on the full set momenta of the $(m+1)$ partons in a non-factorizable way, but are not integrated any further. Any additional terms of type (c) are chosen such that the $\mathcal{X}_3^0$ depends only on m-parton momenta obtained from the phase space mapping, with the consequence that the integration of $(\mathcal{X}_3^0 \cdot X_3^0)$ factorizes, involving only the known integral of X_3^0. For terms of type (d), only the soft momentum is required to be part of the phase space mapping, and thus an integration momentum, while the two hard radiator momenta are kept fixed. It is thus possible [96] for terms of type (d) to choose the momentum mapping such that the analytic integration involves a final-final or initial-final phase space, for which the integrated soft factors are known [53, 54, 82].

With radiator partons i and k in the initial state, the contribution of type (b) reads:

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}^{\mathrm{VS},1,b}_{\mathrm{NNLO}} = \mathcal{N} \sum_{m+1} \mathrm{d}\Phi_{m+1}(k_1,\ldots,k_{m+1};p_1,p_2)\frac{1}{S_{m+1}} \\
\times \sum_j \Bigg[X^0_{ik,j}\, |\mathcal{M}^1_m(\tilde{k}_1,\ldots,\tilde{k}_{j-1},\tilde{k}_{j+1},\ldots,\tilde{k}_{m+1};x_1p_1,x_2p_2)|^2 \\
\times J^{(m)}_m(\tilde{k}_1,\ldots,\tilde{k}_{j-1},\tilde{k}_{j+1},\ldots,\tilde{k}_{m+1}) \\
+\; X^1_{ik,j}\, |\mathcal{M}_m(\tilde{k}_1,\ldots,\tilde{k}_{j-1},\tilde{k}_{j+1},\ldots,\tilde{k}_{m+1};x_1p_1,x_2p_2)|^2 \\
\times J^{(m)}_m(\tilde{k}_1,\ldots,\tilde{k}_{j-1},\tilde{k}_{j+1},\ldots,\tilde{k}_{m+1}) \Bigg], \qquad (2.11)
\end{aligned}
$$

In here, $X^1_{ik,j}$ denotes a one-loop three-parton initial-initial antenna function, which is the only new ingredient. These antenna functions can be obtained by crossing from their final-final counterparts, listed in [49], and have to be integrated over the appropriate phase space according to (2.6).

3 Integration of one-loop antenna functions

The one-loop antenna functions are derived from the interference of one-loop matrix elements with the corresponding Born amplitudes for all $2 \to 2$ processes [50–52] obtained from $\gamma^* \to q\bar{q}g$ (quark-antiquark antenna functions), $\tilde{\chi} \to \tilde{g}gg$ and $\tilde{\chi} \to \tilde{g}q\bar{q}$ (quark-gluon antenna functions) and $H \to ggg$, $H \to gq\bar{q}$ (gluon-gluon antenna functions) by crossing the off-shell current into the final state and two partons into the initial state. We denote a three-particle initial-initial antenna function with partons (i,j) in the initial state and parton k in the final state as $X^0_{ij,k}$ at tree-level and as $X^1_{ij,k}$ at one-loop. The tree-level and one-loop initial-initial antenna functions are summarized in tables 1–3. The usual notation

Quark-antiquark initiated	tree level	one-loop
quark-quark		
$q\bar{q} \to g$	$A^0_{q\bar{q},g}$	$A^1_{q\bar{q},g}$, $\tilde{A}^1_{q\bar{q},g}$, $\hat{A}^1_{q\bar{q},g}$
quark-gluon		
$q\bar{q}' \to \bar{q}'$	$E^0_{q\bar{q}',\bar{q}'}$	$E^1_{q\bar{q}',\bar{q}'}$, $\tilde{E}^1_{q\bar{q}',\bar{q}'}$, $\hat{E}^1_{q\bar{q}',\bar{q}'}$
$q'\bar{q}' \to q$	$E^0_{q'\bar{q}',q}$	$E^1_{q'\bar{q}',q}$, $\tilde{E}^1_{q'\bar{q}',q}$, $\hat{E}^1_{q'\bar{q}',q}$
gluon-gluon		
$q\bar{q} \to g$	$G^0_{q\bar{q},g}$	$G^1_{q\bar{q},g}$, $\tilde{G}^1_{q\bar{q},g}$, $\hat{G}^1_{q\bar{q},g}$

Table 1. List of tree level and one loop three-parton antenna functions for the configurations with a quark-antiquark system in the initial state.

Quark-gluon initiated	tree level	one-loop
quark-quark		
$qg \to q$	$A^0_{qg,q}$	$A^1_{qg,q}$, $\tilde{A}^1_{qg,q}$, $\hat{A}^1_{qg,q}$
quark-gluon		
$qg \to g$	$D^0_{qg,g}$	$D^1_{qg,g}$, $\hat{D}^1_{qg,g}$
gluon-gluon		
$qg \to q$	$G^0_{qg,q}$	$G^1_{qg,q}$, $\tilde{G}^1_{qg,q}$ $\hat{G}^1_{qg,q}$

Table 2. List of tree level and one loop three-parton antenna functions for the configurations with a quark-gluon system in the initial state.

Gluon-gluon initiated	tree level	one-loop
quark-gluon		
$gg \to q$	$D^0_{gg,q}$	$D^1_{gg,q}$, $\hat{D}^1_{gg,q}$
gluon-gluon		
$gg \to g$	$F^0_{gg,g}$	$F^1_{gg,g}$, $\hat{F}^1_{gg,g}$

Table 3. List of tree level and one loop three-parton antenna functions for the configurations with a gluon-gluon system in the initial state.

is used , i.e. $X^1_{ij,k}$ for the leading colour (N) term, $\tilde{X}^1_{ij,k}$ for the subleading ($1/N$) term and $\hat{X}^1_{ij,k}$ for the N_F part.

We start from the unrenormalized NLO three-parton squared matrix elements (normalized to the corresponding two-parton matrix element and divided by the normalization factor $C(\epsilon) = (4\pi)^\epsilon/(8\pi^2)e^{-\gamma_E\epsilon}$) relevant to a particular antenna function, which we denote as $X^{1,U}_{ij,k}$. The antenna function is obtained after renormalization and subtraction of the corresponding tree-level antenna function multiplied by the one-loop correction to the hard radiator pair. Renormalization of the one-loop antenna functions is always carried out in the $\overline{\text{MS}}$-scheme at fixed renormalization scale $\mu^2 = Q^2$. It amounts to a renormalization of the strong coupling constant and (in the case of the quark-gluon and gluon-gluon antenna functions) to a renormalization of the effective operators used to couple an external current to the partonic radiators. In [50–52] it was indeed shown that quark-gluon and gluon-gluon antenna functions can be systematically derived from the effective Lagrangians describing heavy neutralino decay and Higgs boson decay into gluons respectively. The relation between the renormalized and unrenormalized NLO three-parton squared matrix elements is as follows:

$$X_{ij,k}^{1,R} = X_{ij,k}^{1,U} - \frac{b_0}{\epsilon} X_{ij,k}^0 - \frac{\eta_0}{\epsilon} X_{ij,k}^0 \,, \tag{3.1}$$
$$\tilde{X}_{ij,k}^{1,R} = \tilde{X}_{ij,k}^{1,U} \,, \tag{3.2}$$
$$\hat{X}_{ij,k}^{1,R} = \hat{X}_{ij,k}^{1,U} - \frac{b_{0,F}}{\epsilon} X_{ij,k}^0 - \frac{\eta_{0,F}}{\epsilon} X_{ij,k}^0 \,, \tag{3.3}$$

where

$$b_0 = \frac{11}{6} \,, \quad b_{0,F} = -\frac{1}{3} \tag{3.4}$$

are the colour-ordered coefficients of the one-loop QCD β-function:

$$\beta_0 = b_0 N + b_{0,F} N_F \,. \tag{3.5}$$

The renormalization constants for the effective operators are

$$\begin{aligned} &\eta_0 = 0 \,, && \eta_{0,F} = 0 && \text{for } X = A \,, \\ &\eta_0 = b_0 + \frac{3}{2} \,, && \eta_{0,F} = b_{0,F} && \text{for } X = D, E \,, \\ &\eta_0 = 2\, b_0 \,, && \eta_{0,F} = 2\, b_{0,F} && \text{for } X = F, G \,. \end{aligned}$$

The one-loop antenna functions are obtained from the renormalized NLO three-parton squared matrix elements by subtracting from them the product of the tree-level antenna function with the virtual one-loop hard radiator vertex correction [9–13, 49]:

$$X_{ij,k}^1 = X_{ij,k}^{1,R} - \mathcal{X}_2^1 \, X_{ij,k}^0 \,, \tag{3.6}$$
$$\tilde{X}_{ij,k}^1 = \tilde{X}_{ij,k}^{1,R} - \tilde{\mathcal{X}}_2^1 \, X_{ij,k}^0 \,, \tag{3.7}$$
$$\hat{X}_{ij,k}^1 = \hat{X}_{ij,k}^{1,R} - \hat{\mathcal{X}}_2^1 \, X_{i,jk}^0 \,. \tag{3.8}$$

The one-loop corrections to the hard radiator vertex are listed in [49] for $\mathcal{A}_2^1$, $\mathcal{D}_2^1$, $\hat{\mathcal{D}}_2^1$, $\mathcal{F}_2^1$ and $\hat{\mathcal{F}}_2^1$. From these, the remaining functions follow:

$$\tilde{\mathcal{A}}_2^1 = \mathcal{A}_2^1 \,, \qquad \hat{\mathcal{A}}_2^1 = 0 \,, \tag{3.9}$$
$$\tilde{\mathcal{D}}_2^1 = 0 \,, \tag{3.10}$$
$$\mathcal{E}_2^1 = \mathcal{D}_2^1 \qquad \tilde{\mathcal{E}}_2^1 = 0 \,, \qquad \hat{\mathcal{E}}_2^1 = \hat{\mathcal{D}}_2^1 \,, \tag{3.11}$$
$$\tilde{\mathcal{F}}_2^1 = 0 \,, \tag{3.12}$$
$$\mathcal{G}_2^1 = \mathcal{F}_2^1 \qquad \tilde{\mathcal{G}}_2^1 = 0 \,, \qquad \hat{\mathcal{G}}_2^1 = \hat{\mathcal{F}}_2^1 \,. \tag{3.13}$$

The integrated forms of the single real radiation antenna functions are still differential in x_1 and x_2, such that no explicit integration has to be carried out. However, endpoint singularities can occur in either or both of these variables, which have to be regularized dimensionally. This regularization is obtained from the d-dimensional integrated antenna functions (2.6) by expanding the product of the Jacobian factor and the antenna function in distributions using

$$(1-z)^{-1-\epsilon} = -\frac{1}{\epsilon}\,\delta(1-z) + \sum_n \frac{(-\epsilon)^n}{n!}\mathcal{D}_n(1-z)\,, \tag{3.14}$$

with

$$\mathcal{D}_n(1-z) = \left(\frac{\ln^n(1-z)}{1-z}\right)_+ .$$

The expansion is straightforward for the tree-level antenna functions [80], which contain only rational factors in the invariants

$$s_{12} = \frac{q^2}{x_1 x_2}\,, \qquad s_{1j} = -q^2\frac{1-x_2^2}{x_2(x_1+x_2)}\,, \qquad s_{2j} = -q^2\frac{1-x_1^2}{x_1(x_1+x_2)}\,, \tag{3.15}$$

where we denote the pair of initial state partons with indices $(1,2)$ and we refer to the unresolved one with the index j. The one-loop antenna functions contain logarithms and polylogarithms, yielding branch-cuts at the kinematical endpoints, which forbid a direct expansion in distributions. Instead, we start from the unintegrated expression in terms of one-loop master integrals. Only two types of master integrals appear: the one-loop bubble

$$\mathrm{Bub}(p^2) = \left[\frac{(4\pi)^\epsilon}{16\pi^2}\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\frac{i}{\epsilon\,(1-2\epsilon)}\left(-p^2\right)^{-\epsilon} \equiv A_{2,\mathrm{LO}}\left(-p^2\right)^{-\epsilon} . \tag{3.16}$$

and the general one-loop box with one off-shell leg

$$\begin{aligned}\mathrm{Box}(s_{ij},s_{ik}) = {}& \frac{2(1-2\epsilon)}{\epsilon}A_{2,\mathrm{LO}}\frac{1}{s_{ij}s_{ik}} \\ & \Bigg[\left(\frac{s_{ij}s_{ik}}{s_{ij}-s_{ijk}}\right)^{-\epsilon}{}_2F_1\left(-\epsilon,-\epsilon;1-\epsilon;\frac{s_{ijk}-s_{ij}-s_{ik}}{s_{ijk}-s_{ij}}\right) \\ & +\left(\frac{s_{ij}s_{ik}}{s_{ik}-s_{ijk}}\right)^{-\epsilon}{}_2F_1\left(-\epsilon,-\epsilon;1-\epsilon;\frac{s_{ijk}-s_{ij}-s_{ik}}{s_{ijk}-s_{ik}}\right) \\ & -\left(\frac{-s_{ijk}s_{ij}s_{ik}}{(s_{ij}-s_{ijk})(s_{ik}-s_{ijk})}\right)^{-\epsilon}{}_2F_1\left(-\epsilon,-\epsilon;1-\epsilon;\frac{s_{ijk}(s_{ijk}-s_{ij}-s_{ik})}{(s_{ijk}-s_{ij})(s_{ijk}-s_{ik})}\right)\Bigg] .\end{aligned} \tag{3.17}$$

Both integrals appear in all kinematical crossings. The expansion of the terms involving the bubble integral is trivial, while the one-loop box $\mathrm{Box}(s_{ij},s_{ik})$ contains the rational factor $1/(s_{ij}\,s_{ik})$ and appears in the unintegrated antenna functions with further rational prefactors. In terms of the expansion in distributions, one has to distinguish three prefactors: 1, s_{ij}/s_{jk} and s_{ik}/s_{jk}.

In order to analyse the initial-initial kinematical configuration we limit ourselves to the case

$$s_{12} > q^2 > 0, \qquad s_{1j} < 0, \qquad s_{2j} < 0, \tag{3.18}$$

and we study all the possible crossings of the master integrals. For each of the three crossings of the box integral, expansions have to be derived for each of the three prefactors mentioned above.

These expansions proceed by analytic continuation of the hypergeometric functions in the box master integrals to the appropriate region of analyticity, and requiring that the limit $x_i \to 1$ does not result in an argument of the hypergeometric function equal to 1 or infinity (avoiding the branch-cut). In this situation, (3.14) can be applied safely to the coefficients of the hypergeometric function.

Using the following notation

$$\mathrm{M}_{\mathrm{box}}\left(s_{ij}, s_{ik}, \frac{s_{lm}}{s_{pq}}\right) = \frac{2}{C(\epsilon)}\mathcal{J}(x_1, x_2)\frac{s_{lm}}{s_{pq}}\mathcal{R}\left(\mathrm{Box}(s_{ij}, s_{ik})\right), \tag{3.19}$$

$$\mathrm{M}_{\mathrm{bub}}(p^2) = \frac{2}{C(\epsilon)}\mathcal{J}(x_1, x_2)\mathcal{R}\left(\mathrm{Bub}(p^2)\right), \tag{3.20}$$

where $\mathcal{R}$ selects the real part, we list the master integrals with the relevant prefactors in appendix A.

The resulting expressions for the integrated antenna functions $\mathcal{X}^1_{ij,k}$ are very lengthy, such that we only quote one of them, $\hat{\mathcal{D}}_{qg,g}$, in appendix B as an example. Analytic expressions for all of them, as well as for the tree-level antenna functions $\mathcal{X}^0_{ij,k}$ expanded through to $\mathcal{O}(\epsilon^2)$ are attached with the arXiv-submission of this article.

4 Conclusions

In this paper, we extended the antenna subtraction formalism to handle single unresolved radiation at one loop for processes with two partons in the initial state, as required for hadron collider cross sections at NNLO accuracy. The corresponding virtual unresolved subtraction terms consist of tree-level and one-loop antenna functions with both radiators in the initial state. These initial-initial antenna functions are required in unintegrated and integrated form. The unintegrated initial-initial antenna functions are obtained straightforwardly from analytic continuation of the corresponding final-final antennae. The integration of the one-loop antennae over the phase space relevant to the initial-initial configurations requires an expansion in distributions around the kinematical endpoints of the two initial-state momentum fractions, which we performed for all relevant master integrals.

Using the results of this paper in combination with the one-loop antenna functions in the initial-final [82] and final-final [49] case, the NNLO subtraction terms for the one-loop $(n+1)$-parton contribution to n-jet observables at hadron colliders can be constructed and implemented. To accomplish a full NNLO description of n-jet observables at hadron colliders, subtraction terms for the double real radiation $(n+2)$-parton contribution are equally needed. The construction of these subtraction terms in the case of hadronic collisions has been described in [81]. A large fraction of the integrated antenna functions have been derived already [83]. Once the full set of integrated double real radiation antenna functions is completed for the initial-initial case, the antenna subtraction method can be applied to the computation of NNLO corrections to jet production at hadron colliders.

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A Master integrals in initital-initial configuration

The initial-initial one-loop antenna functions can be expressed by the bubble and box master integrals defined in (3.19)–(3.20). They require analytic continuation to the appropriate kinematical region (3.18).

The analytic continuation of the general bubble integral is straightforward

$$\mathrm{M}_{\mathrm{bub}}(p^2) = \frac{2}{C(\epsilon)}\mathcal{J}(x_1,x_2)A_{2,\mathrm{LO}}\mathcal{R}\left(-p^2-i\delta\right)^{-\epsilon}, \tag{A.1}$$

for $p^2>0$. The box integrals read:

$$\begin{aligned}
&(Q^2)^{2+\epsilon}\mathrm{M}_{\mathrm{box}}(s_{1j},s_{2j},1) = -\frac{1}{8\epsilon^4}\delta(1-x_1)\delta(1-x_2) - \frac{1}{2\epsilon^3}\delta(1-x_2)(1+x_1-\mathcal{D}_0(1-x_1))+\\
&\frac{5\pi^2}{96\epsilon^2}\delta(1-x_1)\delta(1-x_2) + \frac{1}{2\epsilon^2}\Bigg((2+2x_1-\mathcal{D}_0(1-x_1))\mathcal{D}_0(1-x_2)-\\
&\delta(1-x_2)\Bigg(2\mathcal{D}_1(1-x_1)-2(1+x_1)\log(1-x_1)-\frac{x_1^2}{1-x_1}\log\left(\frac{1}{4}x_1(1+x_1)^2\right)\Bigg)\Bigg)-\\
&\frac{1}{2\epsilon^2}\frac{1}{(1+x_1)(1+x_2)}\left((1+x_2)^2+2x_1(1+x_2)^2+x_1^2(1+2x_2(1+x_2))\right)-\\
&\frac{1}{24\epsilon}\Bigg(24\Bigg(2(1+x_1)\mathcal{D}_1(1-x_2)+\mathcal{D}_0(1-x_2)\Bigg(-2\mathcal{D}_1(1-x_1)+\\
&2(1+x_1)\log(1-x_1)+\frac{x_1^2}{1-x_1}\log\left(\frac{1}{4}x_1(1+x_1)^2\right)\Bigg)\Bigg)+\delta(1-x_2)\Bigg(-5\pi^2(1+x_1)+\\
&5\pi^2\mathcal{D}_0(1-x_1)-24\mathcal{D}_2(1-x_1)+\frac{6}{1-x_1}\Bigg(-4x_1^2\log(4(1-x_1))\log(1-x_1)+\\
&4\log^2(1-x_1)+4x_1^2\log(1-x_1)\log\left((1+x_1)^2\right)+x_1^2\Bigg(\log^2\left(\frac{4}{(1+x_1)^2}\right)-2\Bigg(\log^2\left(\frac{1}{4}(1+x_1)^2\right)+\\
&\log(x_1)\log\left(\frac{(1+x_1)^2}{4(1-x_1)}\right)\Bigg)\Bigg)\Bigg)-12\frac{x_1^2}{1-x_1}\mathrm{Li}_2(1-x_1)-7\delta(1-x_1)\zeta_3\Bigg)\Bigg)+\\
&\frac{1}{\epsilon}\Bigg(\frac{2}{(1+x_1)(1+x_2)}((1+x_2)^2+2x_1(1+x_2)^2+x_1^2(1+2x_2(1+x_2)))\log(1-x_1)+
\end{aligned}$$

$$
\frac{x_1^2}{(1-x_1)(1-x_2)}\log\left(4x_1(1+x_1)^2\right)-\frac{2x_1^2x_2^2(1+x_1x_2)}{(1-x_1^2)(1-x_2^2)}\log\left(\frac{x_1(1+x_1x_2)}{x_1+x_2}\right)-
$$
$$
\frac{(x_1^2+x_2^2)}{(1-x_1)(1-x_2)}\log(4)-\frac{4x_1^2x_2^2(1+x_1x_2)}{(1-x_1^2)(1-x_2^2)}\log\left(\frac{(x_1+x_2)^2}{(1+x_1)^2}\right)-
$$
$$
\frac{2x_1^2x_2^2(1+x_1x_2)}{(1-x_1^2)(1-x_2^2)}\log\left(\frac{(1+x_1)^2}{(x_1+x_2)(1+x_1x_2)}\right)\Bigg)\Bigg)-
$$
$$
\frac{1}{3840}\Bigg(-13\pi^4\delta(1-x_1)\delta(1-x_2)-160\Bigg(-48\Bigg(\mathcal{D}_1(1-x_1)\mathcal{D}_1(1-x_2)-
$$
$$
(1+x_2)\mathcal{D}_2(1-x_1)-\mathcal{D}_1(1-x_2)\Bigg(2(1+x_1)\log(1-x_1)+\frac{x_1^2}{1-x_1}\log\left(\frac{1}{4}x_1(1+x_1)^2\right)\Bigg)\Bigg)+
$$
$$
\mathcal{D}_0(1-x_2)\Bigg(5\pi^2\mathcal{D}_0(1-x_1)+2\Bigg(-5\pi^2(1+x_1)+24\frac{\log^2(1-x_1)}{1-x_1}+
$$
$$
6\Bigg(-4\mathcal{D}_2(1-x_1)+\frac{x_1^2}{1-x_1}\Bigg(\log^2\left(\frac{4}{(1+x_1)^2}\right)+4\log(1-x_1)\log\left(\frac{(1+x_1)^2}{4(1-x_1)}\right)-
$$
$$
2\Bigg(\log^2\left(\frac{1}{4}(1+x_1)^2\right)+\log(x_1)\log\left(\frac{(1+x_1)^2}{4(1-x_1)}\right)\Bigg)-2\mathrm{Li}_2(1-x_1)\Bigg)\Bigg)\Bigg)\Bigg)+
$$
$$
\delta(1-x_2)\Bigg(10\pi^2\mathcal{D}_1(1-x_1)-16\mathcal{D}_3(1-x_1)-10\pi^2\log(1-x_1)(1+x_1)+
$$
$$
16\log^3(1-x_1)(1+x_1)+\frac{x_1^2}{1-x_1}\Bigg(-\pi^2\log(x_1)+24\log^2(1-x_1)\log\left(\frac{1}{4(1+x_1)^2}\right)-
$$
$$
5\pi^2\log\left(\frac{(1+x_1)^2}{4}\right)+12\log(1-x_1)\log^2\left(\frac{4}{(1+x_1)^2}\right)+2\log^3\left(\frac{4}{(1+x_1)^2}\right)-
$$
$$
24\log(1-x_1)\log^2\left(\frac{1}{4}(1+x_1)^2\right)+4\log^3\left(\frac{1}{4}(1+x_1)^2\right)+48\log^2(1-x_1)\log\left((1+x_1)^2\right)+
$$
$$
6\log(x_1)\log^2\left(\frac{(1+x_1)^2}{4(1-x_1)}\right)+12\log\left(\frac{(1+x_1)^2}{4(1-x_1)}\right)\mathrm{Li}_2(1-x_1)-12\mathrm{Li}_3(1-x_1)\Bigg)+
$$
$$
28(1+x_1-\mathcal{D}_0(1-x_1))\zeta_3\Bigg)\Bigg)\Bigg)-\frac{x_1^2}{2(1-x_1)(1-x_2)}\log^2\left(\frac{4}{(1+x_1)^2}\right)-
$$
$$
\frac{2}{(1+x_1)(1+x_2)}\left((1+x_2)^2+2x_1(1+x_2)^2+x_1^2(1+2x_2(1+x_2))\right)\log^2(1-x_1)+
$$
$$
\frac{x_1^2}{(1-x_1)(1-x_2)}\left(\log(x_1)\log\left(\frac{(1+x_1)^2}{4(1-x_1)}\right)+2\log\left(\frac{4}{(1+x_1)^2}\right)(-\log(1-x_1)-\log(1-x_2))\right)+
$$
$$
\frac{8x_1^2x_2^2(1+x_1x_2)\log(1-x_1)}{(1-x_1^2)(1-x_2^2)}\log\left(\frac{(x_1+x_2)^4}{(1+x_1)^2(1+x_2)^2}\right)-\frac{x_1^2x_2^2(1+x_1x_2)}{(1-x_1^2)(1-x_2^2)}\log^2\left(\frac{(x_1+x_2)^4}{(1+x_1)^2(1+x_2)^2}\right)-
$$
$$
\frac{2}{(1+x_1)(1+x_2)}\left((1+x_2)^2+2x_1(1+x_2)^2+x_1^2(1+2x_2(1+x_2))\right)\log(1-x_2)\log(1-x_1)+
$$
$$
\frac{\log(1-x_1)}{(1-x_1)(1-x_2)}\Bigg(-2x_2^2\log(x_2)+4((x_1^2+x_2^2)\log(4)-2x_1^2\log(1+x_1)-2x_2^2\log(1+x_2))+
$$
$$
4\frac{x_1^2x_2^2(1+x_1x_2)}{(1+x_1)(1+x_2)}\log\left(\frac{(1+x_1)^2(1+x_2)^2}{(x_1+x_2)^2(1+x_1x_2)^2}\right)\Bigg)+\left(\frac{4x_1^2x_2^2(1+x_1x_2)}{(1-x_1^2)(1-x_2^2)}\log(1-x_2)-\right.
$$
$$
\left.\frac{2x_1^2x_2^2(1+x_1x_2)}{(1-x_1^2)(1-x_2^2)}\log\left(\frac{x_2(x_1+x_2)^3}{(1-x_1^2)(1+x_2)^2}\right)\right)\log\left(\frac{x_1(1+x_1x_2)}{x_1+x_2}\right)+
$$
$$
\frac{1}{24(1-x_1^2)(1-x_2^2)}\Bigg(\pi^2\big(5(1-x_2)(1+x_1)(1+x_2)^2+x_1^3(-5-5x_2+6x_2^3)+
$$
$$
x_1^2(-5+x_2(-5+6x_2))\big)+12(1+x_1)(1+x_2)(x_1^2+x_2^2)\log^2(4)+
$$

$$
\begin{aligned}
&48(1+x_1)(1+x_2)\Big(x_1^2\log\Big(\frac{1+x_1}{4}\Big)\log(1+x_1)+x_2^2\log\Big(\frac{1+x_2}{4}\Big)\log(1+x_2)\Big)+\\
&12x_1^2x_2^2(1+x_1x_2)\log^2\Big(\frac{(1+x_1)^2(1+x_2)^2}{(x_1+x_2)^2(1+x_1x_2)^2}\Big)+12x_1^2(1+x_1)(1+x_2)\mathrm{Li}_2(1-x_1)+\\
&12x_2^2\Big((1+x_1)(1+x_2)\mathrm{Li}_2(1-x_2)+2x_1^2(1+x_1x_2)\Big(\mathrm{Li}_2(\frac{(x_1+x_2)^2}{(1+x_1x_2)^2})-\\
&\mathrm{Li}_2(\frac{x_2-x_1^2x_2}{x_1+x_2})-\mathrm{Li}_2(\frac{x_1-x_1x_2^2}{x_1+x_2})\Big)\Big)\Big)+\{x_1\leftrightarrow x_2\}+\mathcal{O}(\epsilon),
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
&(Q^2)^{2+\epsilon}\mathrm{M}_{\mathrm{box}}(s_{12},s_{1j},1)=-\frac{1}{2\epsilon^3}x_1^2\delta(1-x_2)+\frac{1}{2\epsilon^2}x_1^2\Big(2\mathcal{D}_0(1-x_2)+\delta(1-x_2)\log\Big(\frac{4(1-x_1)}{x_1(1+x_1)^2}\Big)\Big)+\\
&\frac{x_1^2}{\epsilon^2}\Big(-2+\frac{1}{1+x_2}-\frac{2x_1x_2^2}{x_1+x_2}\Big)+\frac{1}{24\epsilon}x_1^2\Big(24\Big(-2\mathcal{D}_1(1-x_2)+\mathcal{D}_0(1-x_2)\log\Big(\frac{x_1(1+x_1)^2}{4(1-x_1)}\Big)\Big)\Big)+\\
&\frac{1}{\epsilon}\Big(x_1^2\big(x_2+2x_2^2+x_1(1+2x_2(1+x_2+x_2^2))\big)\frac{\log(1-x_1)}{(1+x_2)(x_1+x_2)}-\frac{x_1^2\log(x_1)}{1-x_2}+\\
&2x_1^2\big(x_2+2x_2^2+x_1(1+2x_2(1+x_2+x_2^2))\big)\frac{\log(1-x_2)}{(1+x_2)(x_1+x_2)}-\frac{x_1^2}{1-x_2}\Big(\log\Big(\frac{(1+x_1)^2}{4}\Big)-\\
&2\frac{x_2^3(1+x_1x_2)}{(x_1+x_2)(1+x_2)}\log\Big(\frac{x_1x_2^2(x_1+x_2)^3}{(1+x_1)(1+x_2)^2}\Big)\Big)\Big)+\\
&\delta(1-x_2)\frac{1}{24\epsilon}x_1^2\Big(\pi^2-6\log\Big(\frac{4(1-x_1)}{(1+x_1)^2}\Big)\log\Big(\frac{4(1-x_1)}{x_1^2(1+x_1)^2}\Big)+12\mathrm{Li}_2(1-x_1)\Big)+\\
&\frac{1}{24}x_1^2\Big(48\Big(\mathcal{D}_2(1-x_2)+\mathcal{D}_1(1-x_2)\log\Big(\frac{4(1-x_1)}{x_1(1+x_1)^2}\Big)\Big)-2\mathcal{D}_0(1-x_2)\Big(\pi^2+12\log(x_1)\log\Big(\frac{4(1-x_1)}{(1+x_1)^2}\Big)-\\
&6\log^2\Big(\frac{4(1-x_1)}{(1+x_1)^2}\Big)+12\mathrm{Li}_2(1-x_1)\Big)+\delta(1-x_2)\Big(2\log(1-x_1)\Big(\log(1-x_1)\log\Big(\frac{64(1-x_1)}{(1+x_1)^6}\Big)-\\
&3\log(x_1)\log\Big(\frac{16(1-x_1)}{(1+x_1)^4}\Big)\Big)-\log\Big(\frac{1-x_1}{x_1}\Big)\Big(\pi^2-6\log^2(4)+6\log\Big(\frac{16}{(1+x_1)^2}\Big)\log\Big((1+x_1)^2\Big)\Big)+\\
&\log\Big(\frac{4}{(1+x_1)^2}\Big)\Big(-\pi^2+2\log^2(4)+2\log\Big(\frac{1}{16}(1+x_1)^2\Big)\log\Big((1+x_1)^2\Big)\Big)+\\
&12\log\Big(\frac{(1+x_1)^2}{4(1-x_1)}\Big)\mathrm{Li}_2(1-x_1)+12\mathrm{Li}_3(1-x_1)+28\zeta_3\Big)\Big)-\frac{2x_1^2}{1-x_2}\log(x_1)\log\Big(\frac{1+x_1}{2}\Big)-\\
&x_1^2\big(x_2+2x_2^2+x_1(1+2x_2(1+x_2+x_2^2))\big)\frac{\log^2(1-x_1)}{2(1+x_2)(x_1+x_2)}-\\
&2x_1^2\big(x_2+2x_2^2+x_1(1+2x_2(1+x_2+x_2^2))\big)\frac{\log^2(1-x_2)}{(1+x_2)(x_1+x_2)}+\log(1-x_1)\Big(\frac{x_1^2\log(x_1)}{1-x_2}-\\
&\frac{x_1^2}{(x_1+x_2)(1-x_2^2)}\Big((1+x_2)(x_1+x_2)\log(4)-2(1+x_2)(x_1+x_2)\log(1+x_1)+\\
&2x_2^3(1+x_1x_2)\log\Big(\frac{x_1x_2^2(x_1+x_2)^3}{(1+x_1)(1+x_2)^2}\Big)\Big)\Big)+\log(1-x_2)\Big(\frac{2x_1^2\log(x_1)}{1-x_2}-2x_1^2\big(x_2+2x_2^2+\\
&x_1(1+2x_2(1+x_2+x_2^2))\big)\frac{\log(1-x_1)}{(1+x_2)(x_1+x_2)}-\frac{2x_1^2}{(x_1+x_2)(1-x_2^2)}\Big((1+x_2)(x_1+x_2)\log(4)-\\
&2(1+x_2)(x_1+x_2)\log(1+x_1)+2x_2^3(1+x_1x_2)\log\Big(\frac{x_1x_2^2(x_1+x_2)^3}{(1+x_1)(1+x_2)^2}\Big)\Big)\Big)-
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{12(x_1+x_2)(1-x_2^2)}x_1^2\Bigg(-\pi^2(1-x_2)\big(x_2+2x_2^2+x_1(1+2x_2(1+x_2+x_2^2))\big)+\\
&6(1+x_2)(x_1+x_2)\log^2(4)+24(1+x_2)(x_1+x_2)\log\left(\frac{1+x_1}{4}\right)\log(1+x_1)-\\
&12x_2^3(1+x_1x_2)\Bigg(\log^2\left(\frac{x_2(x_1+x_2)^3}{(1+x_1)(1+x_2)^2}\right)+\log^2\left(\frac{x_2(x_1+x_2)^3(1-x_1x_2)}{(1+x_1)(1+x_2)^2}\right)+\\
&2\log\left(\frac{x_2(x_1+x_2)^3}{(1+x_1)(1+x_2)^2}\right)\log\left(\frac{x_1(1+x_1x_2)}{x_1+x_2}\right)-\log^2\left(\frac{(x_1+x_2)^2(1-x_1^2x_2^2)}{(1+x_1)(1+x_2)^2}\right)\Bigg)-\\
&12(1+x_2)(x_1+x_2)\mathrm{Li}_2(1-x_1)+24x_2^3(1+x_1x_2)\Bigg(-\mathrm{Li}_2\left(\frac{(1-x_1^2)x_2}{(x_1+x_2)(1-x_1x_2)}\right)+\\
&\mathrm{Li}_2\left(\frac{x_2-x_1^2x_2}{x_1+x_2}\right)+\mathrm{Li}_2\left(\frac{(1-x_1^2)x_2^2}{1-x_1^2x_2^2}\right)\Bigg)\Bigg)+\mathcal{O}(\epsilon),
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
&(Q^2)^{2+\epsilon}\mathrm{M}_{\mathrm{box}}(s_{12},s_{1j},\frac{s_{12}}{s_{2j}})=-\frac{1}{2\epsilon^4}\delta(1-x_1)\delta(1-x_2)-\frac{1}{2\epsilon^3}\delta(1-x_2)(1+x_1-\mathcal{D}_0(1-x_1))+\\
&\frac{1}{\epsilon^3}\delta(1-x_1)(-1-x_2+\mathcal{D}_0(1-x_2))+\frac{1}{\epsilon^2}\mathcal{D}_0(1-x_1)(1+x_2-\mathcal{D}_0(1-x_2))+\frac{1}{\epsilon^2}(1+x_1)\mathcal{D}_0(1-x_2)+\\
&\frac{1}{2\epsilon^2}\delta(1-x_2)\Bigg(-\mathcal{D}_1(1-x_1)+\frac{\log(1-x_1)}{1-x_1}+\frac{x_1^2}{1-x_1}\log\left(\frac{x_1(1+x_1)^2}{4(1-x_1)}\right)\Bigg)-\\
&\frac{1}{24\epsilon^2}\delta(1-x_1)\Bigg(-\pi^2\delta(1-x_2)-24\Bigg(-2\mathcal{D}_1(1-x_2)+2(1+x_2)\log(1-x_2)+\\
&\frac{x_2^2}{1-x_2}\log\left(\frac{1}{2}x_2^2(1+x_2)\right)\Bigg)\Bigg)-\frac{1}{\epsilon^2}\frac{(1+x_2)^2+2x_1(1+x_2)^2+x_1^2(1+2x_2(1+x_2))}{(1+x_1)(1+x_2)}-\\
&\frac{1}{24\epsilon}\Bigg(-\Bigg(24\Bigg(-\Bigg((1+x_2-\mathcal{D}_0(1-x_2))\mathcal{D}_1(1-x_1)+\\
&2(1+x_1-\mathcal{D}_0(1-x_1))\mathcal{D}_1(1-x_2)+\mathcal{D}_0(1-x_2)\Bigg((1+x_1)\log(1-x_1)+\\
&\frac{x_1^2}{1-x_1}\log\left(\frac{1}{4}x_1(1+x_1)^2\right)\Bigg)\Bigg)-2(1+x_2)\mathcal{D}_0(1-x_1)\log(1-x_2)-\\
&\frac{x_2^2}{1-x_2}\mathcal{D}_0(1-x_1)\log\left(\frac{1}{2}x_2^2(1+x_2)\right)\Bigg)-\delta(1-x_2)\Bigg(-\pi^2(1+x_1)+\\
&\pi^2\mathcal{D}_0(1-x_1)+6\frac{\log^2(1-x_1)}{1-x_1}+6\Bigg(-\mathcal{D}_2(1-x_1)-\frac{x_1^2}{1-x_1}\Bigg(\log^2(4)+\\
&2\log(4)\log\left(\frac{1-x_1}{x_1}\right)+\log^2(1-x_1)-2\log(1-x_1)\log(x_1)-2\log\left(4\left(\frac{1-x_1}{x_1}\right)\right)\log\left((1+x_1)^2\right)+\\
&\log^2\left((1+x_1)^2\right)-2\mathrm{Li}_2(1-x_1)\Bigg)\Bigg)\Bigg)\Bigg)+2\delta(1-x_1)\Bigg(-\pi^2(1+x_2)+\pi^2\mathcal{D}_0(1-x_2)-\\
&24\mathcal{D}_2(1-x_2)+24\frac{\log^2(1-x_2)}{1-x_2}-6\frac{x_2}{1-x_2}\Bigg(x_2\Bigg(4\log(1-x_2)\log\left(\frac{2(1-x_2)}{x_2^2(1+x_2)}\right)+\\
&\log^2\left(\frac{1}{2}x_2(1+x_2)\right)+\log(x_2)\log\left(\frac{1}{4}x_2(1-x_2^2)^2\right)\Bigg)\Bigg)-14\delta(1-x_2)\zeta_3\Bigg)\Bigg)+\\
&\frac{1}{\epsilon}\Bigg(((1+x_2)^2+2x_1(1+x_2)^2+x_1^2(1+2x_2(1+x_2)))\frac{1}{(1+x_1)(1+x_2)}\log(1-x_1)+\\
&2((1+x_2)^2+2x_1(1+x_2)^2+x_1^2(1+2x_2(1+x_2)))\frac{1}{(1+x_1)(1+x_2)}\log(1-x_2)+
\end{aligned}
$$

$$
\frac{1}{(1-x_1)(1-x_2)}\left(x_1^2\log\left(\frac{1}{4}x_1(1+x_1)^2\right)+x_2^2\log\left(\frac{1}{2}x_2^2(1+x_2)\right)\right)-
$$
$$
2\frac{x_1^2x_2^2(1+x_1x_2)}{(1-x_1^2)(1-x_2^2)}\log\left(\frac{x_2^2(x_1+x_2)^4}{(1+x_1)(1+x_2)^2(1+x_1x_2)}\right)-\frac{2x_1^2x_2^2(1+x_1x_2)}{(1-x_1^2)(1-x_2^2)}\log\left(\frac{x_1(1+x_1x_2)}{x_1+x_2}\right)\Bigg)+
$$
$$
\frac{1}{24}\Bigg(12(1+x_2)\mathcal{D}_2(1-x_1)-12(4\mathcal{D}_1(1-x_1)\mathcal{D}_1(1-x_2)+\mathcal{D}_0(1-x_2)\mathcal{D}_2(1-x_1))+
$$
$$
48(1+x_1)\mathcal{D}_2(1-x_2)+\delta(1-x_2)\Big(\pi^2\mathcal{D}_1(1-x_1)-2\mathcal{D}_3(1-x_1)\Big)+48\mathcal{D}_1(1-x_2)\log(1-x_1)+
$$
$$
48x_1\mathcal{D}_1(1-x_2)\log(1-x_1)+\frac{48x_1^2}{1-x_1}\mathcal{D}_1(1-x_2)\log\left(\frac{1}{4}x_1(1+x_1)^2\right)+48\mathcal{D}_1(1-x_1)\log(1-x_2)+
$$
$$
48x_2\mathcal{D}_1(1-x_1)\log(1-x_2)+\frac{24x_2^2}{1-x_2}\mathcal{D}_1(1-x_1)\log\left(\frac{1}{2}x_2^2(1+x_2)\right)-\frac{2}{1-x_2}\mathcal{D}_0(1-x_1)\Bigg(\pi^2(1-x_2^2)-
$$
$$
24(1-x_2^2)\log^2(1-x_2)-24x_2^2\log(1-x_2)\log\left(\frac{1}{2}x_2^2(1+x_2)\right)+6x_2^2\Bigg(\log^2\left(\frac{1}{2}x_2(1+x_2)\right)+
$$
$$
\log(x_2)\log\left(\frac{1}{4}x_2(1-x_2^2)^2\right)\Bigg)\Bigg)+\frac{2}{1-x_1}\mathcal{D}_0(1-x_2)\Bigg(-\pi^2(1-x_1^2)+6(1-x_1^2)\log^2(1-x_1)+
$$
$$
12x_1^2\log(1-x_1)\log(x_1)+6x_1^2\log\left(\frac{4(1-x_1)^2}{x_1^2(1+x_1)^2}\right)\log\left(\frac{1}{4}(1+x_1)^2\right)+12x_1^2\mathrm{Li}_2(1-x_1)\Bigg)-
$$
$$
\frac{1}{1-x_2}2\delta(1-x_1)\Bigg(-16(1-x_2^2)\log^3(1-x_2)-24x_2^2\log^2(1-x_2)\log\left(\frac{1}{2}x_2^2(1+x_2)\right)+
$$
$$
2\log(1-x_2)\Bigg(\pi^2(1-x_2^2)+6x_2^2\Bigg(\log^2\left(\frac{1}{2}x_2(1+x_2)\right)+\log(x_2)\log\left(\frac{1}{4}x_2(1-x_2^2)^2\right)\Bigg)\Bigg)+
$$
$$
x_2^2\Bigg(\pi^2\log\left(\frac{1}{2}x_2(1+x_2)\right)-2\log^3\left(\frac{1}{2}x_2(1+x_2)\right)-7\pi^2\log\left(\frac{1}{2}(1-x_2^2)\right)+2\log^3\left(\frac{1}{2}(1-x_2^2)\right)+
$$
$$
7\pi^2\log\left(\frac{1}{2}x_2(1-x_2^2)\right)-2\log^3\left(\frac{1}{2}x_2(1-x_2^2)\right)\Bigg)-28(1-x_2^2)\zeta_3\Bigg)+2\mathcal{D}_0(1-x_1)\Bigg(\pi^2\mathcal{D}_0(1-x_2)-
$$
$$
24\mathcal{D}_2(1-x_2)-14\delta(1-x_2)\zeta_3\Bigg)-\frac{1}{1-x_1}\delta(1-x_2)\Bigg(2x_1^2\log^3(4)+\log(1-x_1)\Bigg(\pi^2+
$$
$$
2\log(1-x_1)(x_1^2\log(64)-(1-x_1^2)\log(1-x_1))-6x_1^2\log(16(1-x_1))\log(x_1)\Bigg)+
$$
$$
6x_1^2\log(4)\log\left(\frac{1-x_1}{x_1}\right)\log(\frac{4}{(1+x_1)^4})+x_1^2\Bigg(\pi^2-6\log^2(4)-6\log(1-x_1)\log\left(\frac{1-x_1}{x_1^2}\right)\Bigg)
$$
$$
\times\log\left((1+x_1)^2\right)-2x_1^2\log^3\left((1+x_1)^2\right)-x_1^2\log\left(4\frac{1-x_1}{x_1}\right)(\pi^2-6\log^2((1+x_1)^2))-
$$
$$
28\zeta_3+4x_1^2\Bigg(3\log\left(\frac{(1+x_1)^2}{4(1-x_1)}\right)\mathrm{Li}_2(1-x_1)+3\mathrm{Li}_3(1-x_1)+7\zeta_3\Bigg)\Bigg)+\frac{1}{120}\delta(1-x_1)(47\pi^4\delta(1-x_2)+
$$
$$
480(\pi^2\mathcal{D}_1(1-x_2)-8\mathcal{D}_3(1-x_2)-14\mathcal{D}_0(1-x_2)\zeta_3))\Bigg)+\frac{2x_1^2}{(1-x_1)(1-x_2)}\log(x_1)\log\left(\frac{1+x_1}{2}\right)-
$$
$$
((1+x_2)^2+2x_1(1+x_2)^2+x_1^2(1+2x_2(1+x_2)))\frac{\log^2(1-x_1)}{2(1+x_1)(1+x_2)}-
$$
$$
2((1+x_2)^2+2x_1(1+x_2)^2+x_1^2(1+2x_2(1+x_2)))\frac{\log^2(1-x_2)}{(1+x_1)(1+x_2)}-
$$
$$
\frac{\log(1-x_1)}{(1-x_1^2)(1-x_2^2)}\Bigg((1+x_1)(1+x_2)\Bigg(x_1^2\log\left(\frac{1}{4}x_1(1+x_1)^2\right)+x_2^2\log\left(\frac{1}{2}x_2^2(1+x_2)\right)\Bigg)-
$$

$$2x_1^2x_2^2(1+x_1x_2)\log\left(\frac{x_2^2(x_1+x_2)^4}{(1+x_1)(1+x_2)^2(1+x_1x_2)}\right)\Bigg)+$$
$$\log(1-x_2)\Bigg(-2\Big((1+x_2)^2+2x_1(1+x_2)^2+x_1^2(1+2x_2(1+x_2))\Big)\frac{\log(1-x_1)}{(1+x_1)(1+x_2)}-$$
$$\frac{2}{(1-x_1^2)(1-x_2^2)}\Bigg((1+x_1)(1+x_2)\Bigg(x_1^2\log\left(\frac{1}{4}x_1(1+x_1)^2\right)+x_2^2\log\left(\frac{1}{2}x_2^2(1+x_2)\right)\Bigg)-$$
$$2x_1^2x_2^2(1+x_1x_2)\log\left(\frac{x_2^2(x_1+x_2)^4}{(1+x_1)(1+x_2)^2(1+x_1x_2)}\right)\Bigg)\Bigg)+\Bigg(\frac{2x_1^2x_2^2(1+x_1x_2)\log(1-x_1)}{(1-x_1^2)(1-x_2^2)}+$$
$$\frac{4x_1^2x_2^2(1+x_1x_2)\log(1-x_2)}{(1-x_1^2)(1-x_2^2)}-\frac{2x_1^2x_2^2(1+x_1x_2)}{(1-x_1^2)(1-x_2^2)}\log\left(\frac{x_2(x_1+x_2)^3}{(1+x_1)(1+x_2)^2}\right)\Bigg)\log\left(\frac{x_1(1+x_1x_2)}{x_1+x_2}\right)+$$
$$\frac{1}{12(1-x_1^2)(1-x_2^2)}\Bigg(\pi^2(1-x_1)(1-x_2)\Big((1+x_2)^2+2x_1(1+x_2)^2+x_1^2(1+2x_2(1+x_2))\Big)+$$
$$6(1+x_1)(1+x_2)\Big(4x_1^2+x_2^2\Big)\log^2(2)-48x_1^2(1+x_1)(1+x_2)\log(2)\log(1+x_1)+$$
$$24x_1^2(1+x_1)(1+x_2)\log^2(1+x_1)+6x_2^2\Bigg(-2(1+x_1)(1+x_2)\log(2)\log(x_2(1+x_2))+$$
$$(1+x_1)(1+x_2)\log^2(x_2(1+x_2))-2x_1^2(1+x_1x_2)\Bigg(\log^2\left(\frac{x_2(x_1+x_2)^3}{(1+x_1)(1+x_2)^2}\right)+$$
$$\log^2\left(\frac{x_2(x_1+x_2)^3(1-x_1x_2)}{(1+x_1)(1+x_2)^2}\right)\Bigg)-(1+x_1)(1+x_2)\log^2\left(\frac{1}{2}(1-x_2^2)\right)+$$
$$2x_1^2(1+x_1x_2)\log^2\left(\frac{(x_1+x_2)^2(1-x_1^2x_2^2)}{(1+x_1)(1+x_2)^2}\right)+(1+x_1)(1+x_2)\log^2\left(\frac{1}{2}(x_2-x_2^3)\right)\Bigg)+$$
$$12x_1^2\Bigg(-(1+x_1)(1+x_2)\mathrm{Li}_2(1-x_1)+2x_2^2(1+x_1x_2)\Bigg(-\mathrm{Li}_2\left(\frac{(1-x_1^2)x_2}{(x_1+x_2)(1-x_1x_2)}\right)+$$
$$\mathrm{Li}_2\left(\frac{x_2-x_1^2x_2}{x_1+x_2}\right)+\mathrm{Li}_2\left(\frac{(1-x_1^2)x_2^2}{1-x_1^2x_2^2}\right)\Bigg)\Bigg)\Bigg)\Bigg)+\mathcal{O}(\epsilon), \tag{A.4}$$

$$(Q^2)^{2+\epsilon}\mathrm{M}_{\mathrm{box}}(s_{12},s_{1j},\frac{s_{1j}}{s_{2j}})=-\frac{x_2^2}{\epsilon^3}\delta(1-x_1)+\frac{x_2^2}{\epsilon^2}\left(\mathcal{D}_0(1-x_1)-\delta(1-x_1)\log\left(\frac{x_2^2(1+x_2)}{2(1-x_2)^2}\right)\right)+$$
$$\frac{x_2^2}{\epsilon^2}\left(-2+\frac{1}{1+x_1}-\frac{2x_1^2x_2}{x_1+x_2}\right)+\frac{1}{12\epsilon}x_2^2\Bigg(-12\Bigg(\mathcal{D}_1(1-x_1)+\mathcal{D}_0(1-x_1)\log\left(\frac{2(1-x_2)^2}{x_2^2(1+x_2)}\right)\Bigg)+$$
$$\delta(1-x_1)\Bigg(\pi^2-6\log^2\left(\frac{x_2(1+x_2)}{2(1-x_2)^2}\right)+6\log\left(\frac{1}{x_2}\right)\log\left(\frac{x_2(1+x_2)^2}{4(1-x_2)^2}\right)\Bigg)\Bigg)+$$
$$\frac{1}{\epsilon}\Bigg(x_2^2\big(x_2+x_1(1+2x_1+2(1+x_1+x_1^2)x_2)\big)\frac{\log(1-x_1)}{(1+x_1)(x_1+x_2)}-\frac{x_2^2}{1-x_1}\Bigg(\log\left(\frac{x_2^2(1+x_2)}{2(-1+x_2)^2}\right)-$$
$$2\frac{x_1^3(1+x_1x_2)}{(1+x_1)(x_1+x_2)}\log\left(\frac{x_1x_2^2(x_1+x_2)^3}{(1+x_1)(1-x_2^2)^2}\right)\Bigg)\Bigg)+$$
$$\frac{1}{12}x_2^2\Bigg(6\mathcal{D}_2(1-x_1)-12\mathcal{D}_1(1-x_1)\log\left(\frac{x_2^2(1+x_2)}{2(1-x_2)^2}\right)-$$
$$\mathcal{D}_0(1-x_1)\Bigg(\pi^2-6\log^2\left(\frac{x_2(1+x_2)}{2(1-x_2)^2}\right)+6\log\left(\frac{1}{x_2}\right)\log\left(\frac{x_2(1+x_2)^2}{4(1-x_2)^2}\right)\Bigg)+$$
$$\delta(1-x_1)\Bigg(-7\pi^2\log\left(\frac{1+x_2}{2(1-x_2)}\right)+2\log^3\left(\frac{1+x_2}{2(1-x_2)}\right)+7\pi^2\log\left(\frac{x_2(1+x_2)}{2(1-x_2)}\right)-$$

$$
\begin{aligned}
&2\log^3\left(\frac{x_2(1+x_2)}{2(1-x_2)}\right)+\pi^2\log\left(\frac{x_2(1+x_2)}{2(1-x_2)^2}\right)-2\log^3\left(\frac{x_2(1+x_2)}{2(1-x_2)^2}\right)+28\zeta_3\Bigg)\Bigg)-\\
&x_2^2\big(x_2+x_1(1+2x_1+2(1+x_1+x_1^2)x_2)\big)\frac{\log^2(1-x_1)}{2(1+x_1)(x_1+x_2)}+\frac{2x_1^3x_2^2(1+x_1x_2)}{(1-x_1^2)(x_1+x_2)}\\
&\times\log\left(\frac{x_1(1+x_1x_2)}{x_1+x_2}\right)\log\left(\frac{x_2(x_1+x_2)^3}{(1+x_1)(1-x_2^2)^2}\right)+\frac{x_2^2}{1-x_1}\log(1-x_1)\Bigg(\log\left(\frac{x_2^2(1+x_2)}{2(1-x_2)^2}\right)-\\
&2\frac{x_1^3(1+x_1x_2)}{(1+x_1)(x_1+x_2)}\log\left(\frac{x_1x_2^2(x_1+x_2)^3}{(1+x_1)(1-x_2^2)^2}\right)\Bigg)\Bigg)-\frac{1}{12(1-x_1^2)(x_1+x_2)}x_2^2\Bigg(-\pi^2(1-x_1)\big(x_2+\\
&x_1(1+2x_1+2(1+x_1+x_1^2)x_2)\big)-6(1+x_1)(x_1+x_2)\log^2\left(\frac{1+x_2}{2-2x_2}\right)+\\
&6(1+x_1)(x_1+x_2)\log^2\left(\frac{x_2(1+x_2)}{2-2x_2}\right)+6\Bigg((1+x_1)(x_1+x_2)\log^2\left(\frac{x_2(1+x_2)}{2(-1+x_2)^2}\right)-\\
&2x_1^3(1+x_1x_2)\Bigg(\log^2\left(\frac{x_2(x_1+x_2)^3}{(1+x_1)(1-x_2^2)^2}\right)+\log^2\left(\frac{x_2(x_1+x_2)^3(1-x_1x_2)}{(1+x_1)(1-x_2^2)^2}\right)-\\
&\log^2\left(\frac{(x_1+x_2)^2(1-x_1^2x_2^2)}{(1+x_1)(1-x_2^2)^2}\right)\Bigg)\Bigg)+24x_1^3(1+x_1x_2)\Bigg(-\mathrm{Li}_2\left(\frac{(1-x_1^2)x_2}{(x_1+x_2)(1-x_1x_2)}\right)+\\
&\mathrm{Li}_2\left(\frac{x_2-x_1^2x_2}{x_1+x_2}\right)+\mathrm{Li}_2\left(\frac{(1-x_1^2)x_2^2}{1-x_1^2x_2^2}\right)\Bigg)\Bigg)+\mathcal{O}(\epsilon),
\end{aligned}
\tag{A.5}
$$

$$
\begin{aligned}
&(Q^2)^{2+\epsilon}\mathrm{M}_{\mathrm{box}}(s_{1j},s_{2j},\frac{s_{1j}}{s_{12}})=-\frac{1}{2\epsilon^3}x_2^2\delta(1-x_1)+\frac{1}{2\epsilon^2}x_2^2\left(2\mathcal{D}_0(1-x_1)+\delta(1-x_1)\log\left(\frac{4(1-x_2)^2}{x_2(1+x_2)^2}\right)\right)-\\
&\frac{1}{\epsilon^2}\frac{x_2^2}{(1+x_1)(x_1+x_2)}\left(x_2+x_1(1+2x_1+2(1+x_1+x_1^2)x_2)\right)-\\
&\frac{x_2^2}{\epsilon}\left(2\mathcal{D}_1(1-x_1)+\mathcal{D}_0(1-x_1)\log\left(\frac{4(1-x_2)^2}{x_2(1+x_2)^2}\right)\right)+\frac{1}{24\epsilon}x_2^2\delta(1-x_1)\Bigg(5\pi^2+6\log^2\left(\frac{4}{(1+x_2)^2}\right)-\\
&12\log^2\left(\frac{1}{4}(1+x_2)^2\right)+12\log\left(\frac{(1-x_2)^2}{x_2}\right)\log\left(\frac{(1+x_2)^2}{4(1-x_2)}\right)-12\mathrm{Li}_2(1-x_2)\Bigg)+\\
&\frac{1}{\epsilon}\Bigg(2x_2^2\big(x_2+x_1(1+2x_1+2(1+x_1+x_1^2)x_2)\big)\frac{\log((1-x_1)(1-x_2))}{(1+x_1)(x_1+x_2)}-\frac{x_2^2}{1-x_1}\log\left(\frac{4x_2}{(1+x_2)^2}\right)-\\
&\frac{4(1+x_1)x_2^2}{1-x_1^2}\log\left(\frac{1+x_2}{2}\right)+\frac{2x_1^3x_2^2(1+x_1x_2)}{(1-x_1^2)(x_1+x_2)}\log\left(\frac{x_1x_2(x_1+x_2)^4}{(1+x_1)^2(1+x_2)^2}\right)\Bigg)+\\
&\frac{1}{24}x_2^2\Bigg(48\mathcal{D}_2(1-x_1)+48\mathcal{D}_1(1-x_1)\log\left(\frac{4(1-x_2)^2}{x_2(1+x_2)^2}\right)+2\mathcal{D}_0(1-x_1)\Bigg(-5\pi^2-6\log^2\left(\frac{4}{(1+x_2)^2}\right)+\\
&24\log(1-x_2)\log\left(\frac{4(1-x_2)}{(1+x_2)^2}\right)+12\log^2\left(\frac{1}{4}(1+x_2)^2\right)+12\log(x_2)\log\left(\frac{(1+x_2)^2}{4(1-x_2)}\right)+\\
&12\mathrm{Li}_2(1-x_2)\Bigg)+\delta(1-x_1)\Bigg(16\log^3(1-x_2)+5\pi^2\log\left(\frac{4}{(1+x_2)^2}\right)-2\log^3\left(\frac{4}{(1+x_2)^2}\right)+\\
&10\pi^2\log\left(\frac{1}{4}(1+x_2)^2\right)-24\log^2(1-x_2)\log\left(\frac{1}{4}(1+x_2)^2\right)-4\log^3\left(\frac{1}{4}(1+x_2)^2\right)-\\
&2\log(1-x_2)\left(5\pi^2+6\log^2\left(\frac{4}{(1+x_2)^2}\right)-12\log^2\left(\frac{1}{4}(1+x_2)^2\right)\right)+\log(x_2)\Bigg(\pi^2-\\
&6\log^2\left(\frac{(1+x_2)^2}{4(1-x_2)}\right)\Bigg)-12\log\left(\frac{(1+x_2)^2}{4(1-x_2)}\right)\mathrm{Li}_2(1-x_2)+12\mathrm{Li}_3(1-x_2)+28\zeta_3\Bigg)\Bigg)-
\end{aligned}
$$

$$\frac{(x_1^3+x_1^3x_2-(1-x_1)x_2^3)\log^2(4)}{(1-x_1)(x_1+x_2)}+\frac{1}{(1-x_1)(x_1+x_2)}2\bigg(2(x_1^3+x_1^3x_2+(-1+x_1)x_2^3)\log^2(2)+$$
$$x_1x_2^2(1+x_2)\log\left(\frac{1}{4}(1+x_1)^2\right)\log((1-x_1)(1-x_2))+\log(1-x_2)\bigg(-x_1(1-x_2)x_2^2\log(4)-$$
$$4x_1^3(1+x_2)\log(1+x_1)+x_1^3(1+x_2)\log\left(4(1+x_1)^2\right)+x_1^3\log\left(\frac{1}{4}(1+x_1)^2\right)\log((1-x_1)(1-x_2))+$$
$$x_1^3x_2\log\left(\frac{1}{4}(1+x_1)^2\right)\log((1-x_1)(1-x_2))-x_1x_2^2\log\left(\frac{1}{16}(1+x_2)^2\right)-$$
$$x_1x_2^3\log\left((1+x_2)^2\right)\bigg)+x_1\log(1-x_1)\bigg(-x_1^2(1+x_1)(1+x_2)\log\left(\frac{1}{4}(1+x_1)^2\right)+$$
$$x_2^2\bigg(x_1\log(16)-4(1+x_1)(1+x_2)\log(1+x_2)+\log\left(4(1+x_2)^2\right)+x_2\log\left(64(1+x_2)^2\right)-$$
$$(1-x_1)x_2\log\left(4x_2(1+x_2)^2\right)\bigg)\bigg)\bigg)-\frac{2x_1^3x_2^2(1+x_1x_2)}{(1-x_1^2)(x_1+x_2)}\bigg(\log\left(\frac{1}{(1-x_2)^2}\right)\log\left(\frac{x_1(1+x_1x_2)}{x_1+x_2}\right)+$$
$$\log\left(\frac{1}{(1-x_1)^2}\right)\log\left(\frac{x_2(1+x_1x_2)}{x_1+x_2}\right)-2\bigg(\log(1-x_2)\log\left(\frac{x_1+x_2}{x_1+x_1^2x_2}\right)+\log(1-x_1)$$
$$\times\log\left(\frac{x_1+x_2}{x_2+x_1x_2^2}\right)\bigg)\bigg)-2x_2^2\big(x_2+x_1(1+2x_1+2(1+x_1+x_1^2)x_2)\big)\frac{\log^2(1-x_1)+\log^2(1-x_2)}{(1+x_1)(x_1+x_2)}+$$
$$\frac{\log((1-x_1)(1-x_2))}{(1-x_1)(x_1+x_2)}\log\left(\frac{4}{(1+x_1)^2}\right)\big(2x_1^3(1+x_2)\log(1-x_2)+2x_1x_2^2(1+x_2)\big)+$$
$$\frac{x_2^2}{2-2x_1}\log^2\left(\frac{4}{(1+x_2)^2}\right)-\frac{x_2^2\log(x_2)}{1-x_1}\log\left(\frac{(1+x_2)^2}{4(1-x_2)}\right)+\frac{2x_1^3x_2^2(1+x_1x_2)}{(1-x_1^2)(x_1+x_2)}\log^2\left(\frac{(x_1+x_2)^4}{(1+x_1)^2(1+x_2)^2}\right)-$$
$$8x_1^3x_2^2(1+x_1x_2)\log((1-x_1)(1-x_2))\frac{1}{(1-x_1^2)(x_1+x_2)}\log\left(\frac{(x_1+x_2)^4}{(1+x_1)^2(1+x_2)^2}\right)+$$
$$\log(1-x_2)\bigg(-4x_2^2\big(x_2+x_1(1+2x_1+2(1+x_1+x_1^2)x_2)\big)\frac{\log(1-x_1)}{(1+x_1)(x_1+x_2)}+\frac{2x_1^3(1+x_2)\log(x_1)}{(1-x_1)(x_1+x_2)}-$$
$$\frac{2}{(1-x_1)(x_1+x_2)}\bigg(-4x_1^3(1+x_2)\log(1+x_1)+x_1^3(1+x_2)\log\left(4x_1(1+x_1)^2\right)+$$
$$x_2^2\bigg(x_1\log(16)-4(x_1+x_2)\log(1+x_2)+x_2\big(x_1\log\left(\frac{4}{(1+x_2)^2}\right)+\log\left(4(1+x_2)^2\right)\big)\bigg)\bigg)-$$
$$4x_1^3x_2^2\frac{(1+x_1x_2)}{(1-x_1^2)(x_1+x_2)}\log\left(\frac{(1+x_1)^2(1+x_2)^2}{(x_1+x_2)^2(1+x_1x_2)^2}\right)\bigg)+\log(1-x_1)\bigg(\frac{2x_1x_2^2(1+x_2)\log(x_2)}{(1-x_1)(x_1+x_2)}-$$
$$2\frac{(1+x_1)}{(1-x_1)(x_1+x_2)}\bigg(x_1^3(1+x_2)\log\left(\frac{4}{(1+x_1)^2}\right)+x_2^2\bigg(x_1\log(16)-4x_1(1+x_2)\log(1+x_2)-$$
$$x_2\log\left(\frac{1}{4}x_2(1+x_2)^2\right)+x_1x_2\log\left(4x_2(1+x_2)^2\right)\bigg)\bigg)-4x_1^3x_2^2\frac{(1+x_1x_2)}{(1-x_1^2)(x_1+x_2)}$$
$$\times\log\left(\frac{(1+x_1)^2(1+x_2)^2}{(x_1+x_2)^2(1+x_1x_2)^2}\right)\bigg)+\frac{2x_1^3x_2^2(1+x_1x_2)}{(1-x_1^2)(x_1+x_2)}\log\left(\frac{x_2(x_1+x_2)^3}{(1-x_1^2)(1-x_2^2)^2}\right)\log\left(\frac{x_1(1+x_1x_2)}{x_1+x_2}\right)+$$
$$\frac{2x_1^3x_2^2(1+x_1x_2)}{(1-x_1^2)(x_1+x_2)}\log\left(\frac{x_2(1+x_1x_2)}{x_1+x_2}\right)\log\left(\frac{x_1(x_1+x_2)^3}{(1-x_1^2)^2(1-x_2^2)}\right)-\frac{1}{12(1-x_1^2)(x_1+x_2)}$$
$$\times x_2^2\bigg(\pi^2\bigg(-5x_2+x_1\big(-5(1+x_2)+x_1(-5+6x_1(1+x_1x_2))\big)\bigg)+12(1+x_1)(x_1+x_2)\log^2(4)+$$
$$48(1+x_1)(x_1+x_2)\log\left(\frac{1+x_2}{4}\right)\log(1+x_2)+12x_1^3(1+x_1x_2)\log^2\left(\frac{(1+x_1)^2(1+x_2)^2}{(x_1+x_2)^2(1+x_1x_2)^2}\right)+$$

$$12(1+x_1)(x_1+x_2)\mathrm{Li}_2(1-x_2)+24x_1^3(1+x_1x_2)\left(\mathrm{Li}_2\left(\frac{(x_1+x_2)^2}{(1+x_1x_2)^2}\right)-\mathrm{Li}_2\left(\frac{x_2-x_1^2x_2}{x_1+x_2}\right)-\right.$$
$$\left.\left.\left.\mathrm{Li}_2\left(\frac{x_1-x_1x_2^2}{x_1+x_2}\right)\right)\right)\right)+\mathcal{O}(\epsilon). \tag{A.6}$$

The remaining box integrals can be obtained from the previous ones by exploiting the symmetry with respect to the transformation $x_1 \leftrightarrow x_2$. More precisely, $\mathrm{M}_{\mathrm{box}}(s_{12}, s_{2j}, 1)$, $\mathrm{M}_{\mathrm{box}}(s_{1j}, s_{2j}, s_{2j}/s_{12})$, $\mathrm{M}_{\mathrm{box}}(s_{12}, s_{2j}, s_{12}/s_{1j})$ and $\mathrm{M}_{\mathrm{box}}(s_{12}, s_{2j}, s_{2j}/s_{1j})$ can be derived from $\mathrm{M}_{\mathrm{box}}(s_{12}, s_{1j}, 1)$, $\mathrm{M}_{\mathrm{box}}(s_{1j}, s_{2j}, s_{1j}/s_{12})$, $\mathrm{M}_{\mathrm{box}}(s_{12}, s_{1j}, s_{12}/s_{2j})$ and $\mathrm{M}_{\mathrm{box}}(s_{12}, s_{1j}, s_{1j}/s_{2j})$ respectively by exchanging $x_1 \leftrightarrow x_2$.

B Example of an integrated antenna function: $\hat{\mathcal{D}}^1_{qg,g}$ antenna

The integrated one-loop antenna functions $\mathcal{X}_{ik,j}$ result in lengthy expressions. As an example of these, the function $\hat{\mathcal{D}}^1_{qg,g}(x_1, x_2)$ reads:

$$(Q^2)^{2\epsilon}\hat{D}^1_{qg,g}(x_1,x_2)=\frac{1}{3\epsilon^3}\delta(1-x_1)\delta(1-x_2)+\frac{1}{6\epsilon^2}\delta(1-x_2)(1+x_1-2\mathcal{D}_0(1-x_1))+$$
$$\frac{1}{3x_2\epsilon^2}\delta(1-x_1)(-1+x_2(2+(-1+x_2)x_2)-x_2\mathcal{D}_0(1-x_2))+$$
$$\frac{1}{36\epsilon}\left(6\left(\frac{1}{(1+x_1)x_2(1+x_2)(x_1+x_2)^3}\left(2x_1^2(2+x_1)+x_1\left(4+x_1(2+x_1)(1+x_1^2)\right)x_2+\right.\right.\right.$$
$$\left(2+x_1^2\left(3+x_1\left(10+x_1(7+2x_1)\right)\right)\right)x_2^2+x_1\left(9+x_1(2+x_1)\left(9+2x_1\left(3+x_1\right)\right)\right)x_2^3+$$
$$\left(3+x_1\left(12+x_1\left(19+2x_1\left(7+x_1\left(5+x_1\right)\right)\right)\right)\right)x_2^4+\left(1+x_1\left(1+x_1\left(4+x_1(2+x_1)\right)\right)\right)$$
$$\left.\times\,2(1+x_1)x_2^5+2x_1(3+2x_1(3+x_1(2+x_1)))x_2^6+2(1+2x_1(1+x_1+x_1^2))x_2^7\right)-\left(2(-\frac{1}{x_2}+\right.$$
$$\left.(2-(1-x_2)x_2))\mathcal{D}_0(1-x_1)+(1+x_1-2\mathcal{D}_0(1-x_1))\mathcal{D}_0(1-x_2)\right)+\delta(1-x_2)\left((1-x_1)+\right.$$
$$\left.\left.\left.2\mathcal{D}_1(1-x_1)-(1+x_1)\log(1-x_1)+\frac{(1+x_1^2)}{1-x_1}\log\left(\frac{2}{1+x_1}\right)\right)\right)\right)-$$
$$\delta(1-x_1)\left(-\left(-\pi^2\delta(1-x_2)-3(x_2-4\mathcal{D}_1(1-x_2))-12(-\frac{1}{x_2}+\right.\right.$$
$$\left.\left.\left.(2-(1-x_2)x_2))\log(1-x_2)\right)-12\frac{(1-(1-x_2)x_2)^2}{(1-x_2)x_2}\log\left(\frac{2}{1+x_2}\right)\right)\right)+$$
$$\frac{1}{72}\left(2\pi^2\delta(1-x_2)\mathcal{D}_0(1-x_1)+\frac{24(-1+x_2(2-(1-x_2)x_2))\mathcal{D}_1(1-x_1)}{x_2}+12(1+x_1)\mathcal{D}_1(1-x_2)-\right.$$
$$24(\mathcal{D}_0(1-x_2)\mathcal{D}_1(1-x_1)+\mathcal{D}_0(1-x_1)\mathcal{D}_1(1-x_2))-12\delta(1-x_2)\mathcal{D}_2(1-x_1)+$$
$$2\delta(1-x_1)\left(\pi^2\mathcal{D}_0(1-x_2)-6\mathcal{D}_2(1-x_2)\right)-\frac{12}{1-x_1}\mathcal{D}_0(1-x_2)\left((1-x_1)^2+x_1^2\log(2(1-x_1))+\right.$$
$$\left.\log\left(\frac{2}{1-x_1}\right)+(1+x_1^2)\log\left(\frac{1}{1+x_1}\right)\right)-\frac{1}{1-x_1}\delta(1-x_2)\left(\pi^2(1-x_1^2)+6(1+x_1^2)\log^2(2)+\right.$$
$$6(\log(4)+x_1^2\log(4(1-x_1)))\log(1-x_1)-6\log^2(1-x_1)+12\log(2(1-x_1))\left((1-x_1)^2+\right.$$

$$
\begin{aligned}
&(1+x_1^2)\log\left(\frac{1}{1+x_1}\right)\Bigg)+6\log\left(\frac{1}{1+x_1}\right)\Bigg(2(1-x_1)^2+(1+x_1^2)\log\left(\frac{1}{1+x_1}\right)\Bigg)\Bigg)+\\
&\frac{12}{x_2}\mathcal{D}_0(1-x_1)\Bigg(x_2^2+2(-1+x_2(2-(1-x_2)x_2))\log(1-x_2)\Bigg)-\\
&\frac{24\mathcal{D}_0(1-x_1)}{(1-x_2)x_2}(1-(1-x_2)x_2)^2\log\left(\frac{2}{1+x_2}\right)-\frac{1}{(1-x_2)x_2}2\delta(1-x_1)\Bigg(-6(1-x_2)(-1+x_2(2-\\
&(1-x_2)x_2))\log^2(1-x_2)-(-1+x_2)\Bigg(6+2x_2(-3+7x_2)+\pi^2(-1+x_2(2-(1-x_2)x_2))-\\
&3x_2^2\log\left(\frac{4}{(1+x_2)^2}\right)\Bigg)+6(1-(1-x_2)x_2)^2\log^2\left(\frac{2}{1+x_2}\right)+3\log(1-x_2)\Bigg(-(1-x_2)x_2^2+\\
&4(1-(1-x_2)x_2)^2\log\left(\frac{2}{1+x_2}\right)\Bigg)\Bigg)\Bigg)+\frac{1}{(1-x_1^2)x_2(x_1+x_2)^3(1-x_2^2)}12\Bigg(x_2\Bigg(-2x_2+x_2^3+\\
&x_2^6+x_1^6\Bigg(1+x_2-2x_2^5\Bigg)-x_1^2(1-x_2)x_2(-1+x_2(1+x_2)(2+5x_2))-x_1(1-x_2)x_2^2(5+\\
&x_2(6+x_2(4+x_2)))+x_1^4\big(-1+x_2-6x_2^5+x_2^3(6+\log(8))\big)-x_1^3\big(-1+x_2(3+x_2+x_2^2-\\
&6x_2^3+2x_2^5-x_2\log(8))\big)+x_1^5\big(-1+x_2(2-10x_2^3+x_2(9+\log(8)))\big)\Bigg)+\Bigg(x_1^6x_2(1+x_2-2x_2^5)+\\
&x_1^5x_2\big(1+x_2(4+3x_2-4x_2^3-4x_2^5)\big)-(1-x_2)x_2^2\big(2+x_2^2(3+2(x_2+x_2^3))\big)-\\
&x_1^3(1-x_2)^2\Bigg(-2+x_2\big(-3+x_2(4+x_2(7+2x_2+4x_2^2))\big)\Bigg)+x_1^2\Bigg(-4+x_2\Bigg(6+x_2\big(-5+\\
&x_2(-6+x_2(2+x_2+6x_2^3))\big)\Bigg)\Bigg)+x_1x_2\Bigg(-4+x_2\big(6+x_2(-11+x_2^2(7+2x_2(-2+x_2(2+x_2))))\big)\Bigg)+\\
&x_1^4\Bigg(2+x_2\Bigg(-3+x_2\big(4+x_2\big(8-x_2(7+2x_2(-1+x_2+2x_2^3))\big)\big)\Bigg)\Bigg)\Bigg)\log(1-x_1)+\\
&x_1^2x_2\Bigg(x_2^4+x_1^4(1+x_2)+x_1^3(1+x_2)(1+3x_2)+x_1x_2^2(3+4x_2)+3x_1^2\Bigg(x_2+x_2^3\Bigg)\Bigg)\log\left(\frac{1}{1+x_1}\right)+\\
&x_2\Bigg(x_1^3(1+x_1)+x_1^2(3+4x_1)x_2+3x_1(1+2x_1)\left(1+x_1^2\right)x_2^2+(1+2x_1)^2x_2^3+\left(1+x_1+x_1^3\right)x_2^4\Bigg)\\
&\times\log\left(\frac{2}{1+x_1}\right)+x_1^4x_2^2\log\left(\frac{16}{1+x_1}\right)+x_1^2x_2\left(x_1^3(1+x_1)+x_1^3(4+x_1)x_2+x_2^4\right)\log(2(1-x_2))-\\
&\Bigg(-2x_1^6x_2^6+x_1^5x_2^3(3-4(x_2^2+x_2^4))-(1-x_2)x_2^2\big(2+x_2^2(3+2(x_2+x_2^3))\big)-\\
&x_1^3(1-x_2)^2\Bigg(-2+x_2\big(-3+x_2(4+x_2(7+2x_2+4x_2^2))\big)\Bigg)+x_1^2\Bigg(-4+x_2\big(6+x_2(-5+2x_2(-3+\\
&x_2+3x_2^4))\big)\Bigg)+x_1x_2\Bigg(-4+x_2\big(6+x_2\big(-11+x_2^2(7+2x_2(-2+x_2(2+x_2)))\big)\big)\Bigg)+\\
&x_1^4\Bigg(2+x_2\Bigg(-3+x_2\big(4+x_2\big(8-x_2(7+2x_2(-1+x_2+2x_2^3))\big)\big)\Bigg)\Bigg)\Bigg)\log(1-x_2)+\\
&2\Bigg(x_1^4(1+x_2)\left(1+(-1+x_2)x_2\right)^2+x_2^3(1+x_2)(1+(-1+x_2)x_2)^2+x_1x_2^2(1+x_2)(3+x_2)\big(1-\\
&(1-x_2)x_2\big)^2+3x_1^2x_2\left(1+x_2^3\right)^2+x_1^3\left(1+x_2\left(2+x_2\left(-2+4x_2-2x_2^3+3x_2^4\right)\right)\right)\Bigg)\log\left(\frac{2}{1+x_2}\right)+\\
&4x_1^3x_2^4\log\left(\frac{4}{1+x_2}\right)-2(1+x_1x_2)\Bigg(x_2^2+x_1^5x_2^5+
\end{aligned}
$$

$$
\begin{aligned}
&x_1x_2(2+x_2^2)+x_1^3x_2^3(3-x_2^2+2x_2^4)+x_1^2(2-x_2^2+3x_2^4)+x_1^4(x_2^4+2x_2^6)\Big)\log\left(\frac{(1+x_1)(1+x_2)}{(x_1+x_2)^2}\right)\Big)-\\
&\frac{1}{(1-x_1^2)x_2(x_1+x_2)^3(1-x_2^2)}24\Big(2x_1^6x_2^6+x_1^5x_2^3(-3+4(x_2^2+x_2^4))+(1-x_2)x_2^2(2+x_2^2(3+2(x_2+x_2^3)))+\\
&x_1^3(1-x_2)^2\Big(-2+x_2(-3+x_2(4+x_2(7+2x_2+4x_2^2)))\Big)+x_1^2\Big(4+x_2(-6+x_2(5-2x_2(-3+\\
&x_2+3x_2^4)))\Big)+x_1x_2(4-x_2(6+x_2(-11+x_2^2(7+2x_2(-2+x_2(2+x_2))))))+x_1^4\Big(-2+x_2\Big(3+\\
&x_2(-4+x_2(-8+x_2(7+2x_2(-1+x_2+2x_2^3))))\Big)\Big)\Big)\log(1-x_2)-8\delta(1-x_1)\delta(1-x_2)\zeta_3\Big)+\mathcal{O}(\epsilon).
\end{aligned}
\tag{B.1}
$$

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